Exam. Code : 211003

Subject Code : 3849

M.Sc. (Mathematics) 3rd Semester

TOPOLOGY—I

Paper : MATH-572

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **two** questions from each Unit. All questions carry **10** marks each.

UNIT—I

1. If X is any set, then prove that the collection of all one point subsets of X is a basis for the discrete topology on X.

2. Prove the following :

   (i) $\overline{A}$ is the smallest closed set containing A.

   (ii) $\overline{\overline{A}} = \overline{A}$.

   (iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

3. Prove that, in a metric space the concept of 2^{nd} countability, separability, and Lindelof are all equivalent.

4. Let X be a set, and u : $\wp(X) \to \wp(X)$ a map with the properties :

   (i) $u(\phi) = \phi$.

   (ii) $A \subset u(A)$ for each A.

   (iii) $u^\circ u(A) = u(A)$ for each A.

   (iv) $u(A \cup B) = u(A) \cup u(B)$ for each A, B.

   Then prove that the family $\mathfrak{I} = \{\mathcal{C}(u(A)) \mid A \in \wp(x)\}$, where $\mathcal{C}(u(A))$ denotes the complement of u(A) is a topology, and with $\mathfrak{I}$, $\overline{A} = u(A)$ for each A.

UNIT—II

5. Prove that a subspace of a subspace is a subspace of the entire space.

6. Let X be a locally connected space. If Y is an open subspace of X, then prove that each component of Y is open in X and hence in particular each component of X is open.

7. Prove that the set of real numbers is connected.

8. Let X be a topological space and A a connected subspace of X. If B is a subspace of X such that $A \subset B \subset \overline{A}$, then, prove that B is connected and hence $\overline{A}$ is connected.

UNIT—III

9. Let X, Y be topological spaces and $f : X \to Y$ a map then prove that the following statements are equivalent :

   (i) f is continuous.

   (ii) $f(\overline{A}) \subset \overline{f(A)}, \forall A \subset X$.

   (iii) $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}), \forall B \subset Y$.

10. Let X be a topological space and $Y \subset X$. Prove that relative topology $\Im_Y$ on Y is the smallest topology on Y for which the inclusion map $i : Y \to X$ is continuous.

11. Let $f : X \to Y$ and $g : Y \to X$ be continous and such that both $g \circ f = 1_X$ and $f \circ g = 1_Y$. Prove that f is a homeomorphism and $g = f^{-1}$.

12. Prove that $f : X \to Y$ is closed map iff $\overline{f(A)} \subset f(\overline{A})$ for each set $A \subset X$.

UNIT—IV

13. Let $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ be any family of spaces, and $f : X \to \Pi_\alpha Y_\alpha$ mapping. Then f is continuous if and only if $p_\beta \circ f$ is continuous for each $\beta \in \mathcal{A}$.

14. In the space $\Pi_\alpha\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ if $A_\alpha \subset Y_\alpha$ for each $\alpha \in \mathcal{A}$, prove that $\overline{\Pi_\alpha A_\alpha} = \Pi_\alpha \overline{A_\alpha}$.

15. Define the quotient space. Prove that if Y is a quotient space of X and Z is a quotient space of Y then Z is homeomorphic to a quotient space of X.

16. If β is a base for the topology of X and $\mathcal{C}$ is a base for the topology Y, then the collection $\mathcal{D} = \{B \times C : B \in \beta, C \in \mathcal{C}\}$ is a base for the topology on $X \times Y$.

UNIT—V

17. Prove that every compact Hausdorff space is normal.

18. Prove that every normal space is regular but converse is not true.

19. State and Prove Urysohn's lemma.

20. Prove that the following three statements are equivalent :

    (i) Y is regular.

    (ii) For each $y \in Y$ and neighbourhood U of y, there exist a neighbourhood V of y with $y \in V \subset \overline{V} \subset U$.

    (iii) For each $y \in Y$ and a closed set A not containing y, there exist a neighbourhood V of y with, $\overline{V} \cap A = \phi$.